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MICROMECHANICAL MODELLING OF ANISOTROPIC NON-LINEAR ELASTICITY OF GRANULAR MEDIUM

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Abstract—In this paper a macroscopic elastic model is derived from a microscopic Hertz–Mindlin elastic contact law using an homogenisation technique. The results obtained in the case of an initially isotropic granular medium submitted to an isotropic stress state are first presented. The influence of the static and kinematic internal variables defined in the homogenisation approach is then discussed. Extensions to the cases of an isotropic granular medium submitted to an anisotropic stress state and of an anisotropic medium loaded isotropically are analysed and discussed. As often as possible, comparisons with experiments or numerical simulations are considered. Copyright © 1996 Elsevier Science Ltd.

1. HOMOGENIZATION TECHNIQUE

The behaviour of granular materials is complex, even the reversible part is not at present clearly established. The reason of this complex behaviour lies firstly in the non-linear local behaviour at the contact point and secondly in the complex evolution of the structure, even at a very low loading level (in particular by creation and loss of contact). The only way to take into account the local complex phenomena is to use an homogenisation technique to define the global behaviour from local phenomena. Such techniques have been used by different authors in previous studies (Koenders, 1987; Walton, 1987; Sidoroff *et al.*, 1992; Chang *et al.*, 1994; Cambou *et al.*, 1995) who have proposed a way which seems to allow homogenisation to be achieved in realistic terms.

The limits of applicability of the proposed analysis are those of the homogenisation theory. The material should present a sufficient homogenity; in particular the behaviour of materials with local instabilities (on singular points or surfaces) cannot reasonably be described through the considered approach.

The global variables are the stress and strain tensors σ_{ij} and ε_{ij} and the following local variables are considered (Cambou and Sidoroff, 1985):

Contact forces: $\mathbf{F}_i$.

Relative displacements of contact points between particles: $\mathbf{U}_i$.

These variables are very complex stochastic ones for which a first approximation description is proposed considering the mean values defined in a solid angle $d\Omega$ for each contact orientation $\mathbf{n}$ (Fig. 1): $\bar{\mathbf{F}}(\mathbf{n})$, $\bar{\mathbf{U}}(\mathbf{n})$.

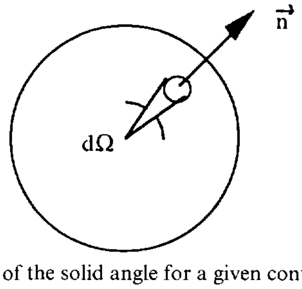


Fig. 1. Definition of the solid angle for a given contact orientation **n**.

The considered medium is composed of spherical particles in a non-regular array whose average diameter is $\bar{D}$. A measure of the geometry arrangement is defined by the contact distribution $P(\mathbf{n})$ which satisfies:

$$\oint_{\text{Unit Sphere}} P(\mathbf{n})\, \mathrm{d}\Omega = 1. \tag{1}$$

For an isotropic medium:

$$P(\mathbf{n}) = \frac{1}{4\pi}. \tag{2}$$

The local variables which are considered in the homogenisation technique are (Cambou *et al.*, 1995):

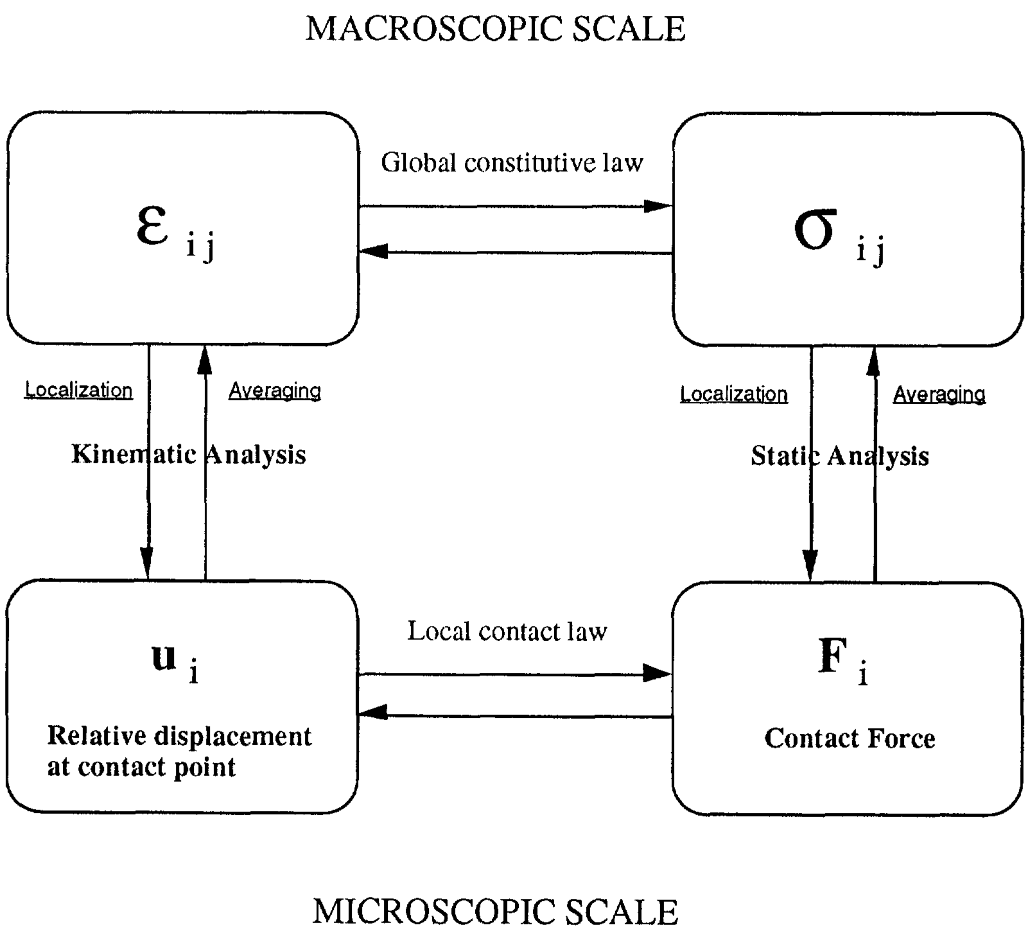
$$\text{Static variable:} \qquad \mathbf{f(n)} = P(\mathbf{n})\bar{\mathbf{F}}(\mathbf{n})\frac{4\pi N\bar{D}}{3} \tag{3}$$

$$\text{Kinematic variable:} \quad \mathbf{u(n)} = \bar{\mathbf{U}}(\mathbf{n})\frac{3}{4\pi\bar{D}} \tag{4}$$

where $\bar{D}$ = mean diameter of particles, and N = number of contacts per unit volume.

According to these definitions, $\mathbf{f(n)}$ is homogenous to a stress and $\mathbf{u(n)}$ to a strain. Both the medium fabric [through $P(\mathbf{n})$] and the force distribution [through $\bar{\mathbf{F}}(\mathbf{n})$] intervene in the static variable $\mathbf{f(n)}$. The homogenisation technique requires the definition of different operators (localisation and averaging) described in Table 1.

Table 1. Definition of the different operators of localisation and averaging



Different hypotheses can be considered for the localisation and averaging operators. Different homogenisation processes have previously been analysed (Cambou *et al.*, 1995) and compared. It can be noted that an homogenisation process is defined by a localisation and an averaging operator. Three approaches are considered here:

Process 0—Voigt type process: $\boldsymbol{\varepsilon} \xrightarrow{L_0^k} \mathbf{u}(\mathbf{n})$ and $\mathbf{f}(\mathbf{n}) \xrightarrow{A_0^s} \boldsymbol{\sigma}$,

Process 1—Static localisation process: $\boldsymbol{\sigma} \xrightarrow{L_1^s \text{ or } L_1^{se}} \mathbf{f}(\mathbf{n})$ and $\mathbf{u}(\mathbf{n}) \xrightarrow{A_1^k \text{ or } A_1^{ke}} \boldsymbol{\varepsilon}$,

Process 2—Second kinematic localisation process: $\boldsymbol{\varepsilon} \xrightarrow{L_2^k \text{ or } L_2^{ke}} \mathbf{u}(\mathbf{n})$ and $\mathbf{f}(\mathbf{n}) \xrightarrow{A_2^s = A_0^s} \boldsymbol{\sigma}$.

In each of these schemes, A stands for averaging operator, L for localisation, s for static analysis, k for kinematic analysis, e indicates whether the medium can be considered as isotropic or anisotropic (the medium anisotropy is then described by a second-order deviatoric tensor **e**), finally the number 0, 1, 2 corresponds to the process that uses these operators.

The derivation of all the operators used in the three processes is based on two relationships:

- The well known averaging operator (Love, 1927):

$$\sigma_{ij} = \frac{1}{V} \sum_{\text{contacts}} F_i l_j. \tag{5}$$

- The balance between the macroscopic work $\sigma : \varepsilon$ and the integral overall contact direction **n** of microscopic work $\mathbf{f}(\mathbf{n}) \cdot \mathbf{u}(\mathbf{n})$:

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \oint \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}\Omega. \tag{6}$$

Process 0

A_0^s is infered from eqn (5) introducing the static variable $\mathbf{f}(\mathbf{n})$

A_0^s
$$\boldsymbol{\sigma} = \frac{3}{4\pi} \oint_{\text{Unit Sphere}} \mathbf{f} \otimes \mathbf{n} \, \mathrm{d}\Omega. \tag{7}$$

L_0^k is the simplest operator satisfying the balance of works

L_0^k
$$\mathbf{u}(\mathbf{n}) = \frac{3}{4\pi} \boldsymbol{\varepsilon} \mathbf{n}. \tag{8}$$

It is noticeable that this localisation operator corresponds to the classical one used in Voigt's homogenisation in continuum mechanics. Therefore Process 0 will also be called Voigt type homogenisation.

Process 1

Resulting from the representation theorem (Spencer, 1987), the localisation operators L_1^s and L_1^{se} are isotropic functions of $\mathbf{n}$, $\boldsymbol{\sigma}$, $\mathbf{e}$, linear in $\boldsymbol{\sigma}$ and $\mathbf{e}$, and odd in $\mathbf{n}$. They also have to be consistent with A_0^s whatever the stress tensor $\boldsymbol{\sigma}$ is

L_1^s $$\mathbf{f}(\mathbf{n}) = \mu\boldsymbol{\sigma}\mathbf{n} + \frac{1-\mu}{2}(5\mathbf{n}\boldsymbol{\sigma}\mathbf{n} - \operatorname{tr}\boldsymbol{\sigma})\mathbf{n} \quad (9)$$

L_1^{se} $$\mathbf{f}(\mathbf{n}) = \mu\boldsymbol{\sigma}\mathbf{n} + \frac{1-\mu}{2}(5\mathbf{n}\boldsymbol{\sigma}\mathbf{n} - \operatorname{tr}\boldsymbol{\sigma})\mathbf{n} + \operatorname{tr}\boldsymbol{\sigma}\left[(\mathbf{nen})\mathbf{n} - \frac{2}{5}\mathbf{en}\right]. \quad (10)$$

The two averaging operators A_1^k and A_1^{ke} are deduced from L_1^s and L_1^{se} considering the balance of macroscopic and microscopic work in eqn (6):

A_1^k $$\boldsymbol{\varepsilon} = \oint_{\text{Unit Sphere}} \left[\mu\mathbf{u} \otimes \mathbf{n} + \frac{1-\mu}{2}[5\mathbf{n} \otimes \mathbf{n} - \delta]\mathbf{u} \cdot \mathbf{n}\right] d\Omega \quad (11)$$

A_1^{ke} $$\boldsymbol{\varepsilon} = \oint_{\text{Unit Sphere}} \left[\mu\mathbf{u} \otimes \mathbf{n} + \frac{1-\mu}{2}[5\mathbf{n} \otimes \mathbf{n} - \delta]\mathbf{u} \cdot \mathbf{n} + \delta\left[(\mathbf{nen})\mathbf{u} \cdot \mathbf{n} - \frac{2}{5}(\mathbf{uen})\right]\right] d\Omega. \quad (12)$$

Process 2

As for Process 1, the localisation operators are resulting from the application of the general representation theorem (L_2^k and L_2^{ke} are now isotropic functions of $\mathbf{n}$, $\boldsymbol{\varepsilon}$, $\mathbf{e}$, linear in $\boldsymbol{\varepsilon}$ and $\mathbf{e}$, and odd in $\mathbf{n}$) and they must verify A_1^k and A_1^{ke} whatever the strain tensor is:

L_2^k $$\mathbf{u}(\mathbf{n}) = \frac{3}{4\pi}\left\{\left[1 + b\left(\frac{3\mu}{5} - 1\right)\right]\boldsymbol{\varepsilon}\mathbf{n} + b\left[\mathbf{n}\boldsymbol{\varepsilon}\mathbf{n} - \frac{\mu}{5}\operatorname{tr}\boldsymbol{\varepsilon}\right]\mathbf{n}\right\} \quad (13)$$

L_2^{ke} $$\mathbf{u}(\mathbf{n}) = \frac{3}{4\pi}\left\{\begin{array}{l}\left[1 + b\left(\frac{3\mu}{5} - 1\right)\right]\boldsymbol{\varepsilon}\mathbf{n} + b\left[\mathbf{n}\boldsymbol{\varepsilon}\mathbf{n} - \frac{\mu}{5}\operatorname{tr}\boldsymbol{\varepsilon}\right]\mathbf{n} \\ + c\left[\begin{array}{l}2\operatorname{tr}(\boldsymbol{\varepsilon}\mathbf{e})\mathbf{n} - 20(\mathbf{ne}\boldsymbol{\varepsilon}\mathbf{n})\mathbf{n} + 35(\mathbf{nen})(\mathbf{n}\boldsymbol{\varepsilon}\mathbf{n}) \\ + 6(\mathbf{e}\boldsymbol{\varepsilon})\mathbf{n} - 6(\boldsymbol{\varepsilon}\mathbf{e})\mathbf{n} - 15(\mathbf{nen})\boldsymbol{\varepsilon}\mathbf{n} - 5\operatorname{tr}\boldsymbol{\varepsilon}\mathbf{en} + 15(\mathbf{n}\boldsymbol{\varepsilon}\mathbf{n})\mathbf{en}\end{array}\right]\end{array}\right\}. \quad (14)$$

As the averaging operator A_0^s already exists, the averaging operator A_2^s has not to be rebuilt, A_2^s is equal to that of Process 0

A_2^s $$\boldsymbol{\sigma} = \frac{3}{4\pi}\oint_{\text{Unit Sphere}} \mathbf{f} \otimes \mathbf{n}\, d\Omega. \quad (15)$$

As a conclusion of this section, it must be stressed that only three parameters have been introduced to describe the distributions of contact forces and relative displacements of any granular medium (the stress history is also taken into account): these parameters μ, b, $\mathbf{e}$ have some physical meanings which are proposed hereafter.

Variable μ has been used to define L_1^s, it has a great influence on the orientation of contact forces. In particular for $\mu = 0$ $\mathbf{f}$ is colinear with $\mathbf{n}$ and for $\mu = 1$ $\mathbf{f}$ is colinear with $\boldsymbol{\sigma}\mathbf{n}$. Of particular interest is the decomposition of the stress tensor $\boldsymbol{\sigma}$ and its deviatoric part $\mathbf{s}$ in two parts $\boldsymbol{\sigma}^N$, $\mathbf{s}^N$ and $\boldsymbol{\sigma}^T$, $\mathbf{s}^T$ respectively, resulting from the normal and tangential components of $\mathbf{f}$. It can be easily shown (Sidoroff *et al.*, 1992) that:

$$\mathbf{s}^N = \left(1 - \frac{3\mu}{5}\right)\mathbf{s}. \quad (16)$$

The parameter μ thus also represents the fraction of the deviatoric stress tensor supported by the normal components of contact forces (from 2/5 for $\mu = 1$ to 1 for $\mu = 0$).

b has appeared in addition to μ in L_2^k. It has been shown by Cambou (1993) that b is linked to the local rotation of particles and to the possible creation and loss of contact in the granular array. It must be stressed that the relative displacement $\mathbf{u}(\mathbf{n})$ has two different physical origins: the displacement of the centres of particles and the rotations of these particles. $b = 0$ corresponds to the case of no possible rotation. As a matter of fact, Process 0 is a particular case of Process 2 with $b = 0$.

$\mathbf{e}$ is a deviatoric tensor characterising the internal anisotropy of the granular medium. This internal anisotropy will lead to anisotropical distributions of local variables, for instance:

- the contact distribution,
- the contact force distribution,
- the relative contact displacement distribution.

2. NON-LINEAR ELASTIC MODEL

To complete the homogenisation process, it is necessary to define the local contact law. The Hertz–Mindlin model is considered, which can be written in an incremental form (Mindlin and Deresiewicz, 1953):

$$\dot{\mathbf{F}}_n = \left(\frac{\sqrt{3R}G_m}{1-\nu_m}\right)^{2/3} (F_n)^{1/3}\dot{\mathbf{U}}_n \tag{17}$$

$$\dot{\mathbf{F}}_t = \left(\frac{\sqrt{3R}G_m}{1-\nu_m}\right)^{2/3} \frac{2(1-\nu_m)}{2-\nu_m}\left(F_n - \frac{F_t}{\tan\Phi}\right)^{1/3}\dot{\mathbf{U}}_t \tag{18}$$

where G_m = shear modulus of the particle material, ν_m = Poisson's ratio of the particle material. At this point it is possible to deduce the elastic constants of a granular medium from the local contact law (G_m, ν_m) and the internal variables of the medium defined in the localisation and averaging operators (μ, or μ and b, $\mathbf{e}$ in the anisotropic case). Three homogenisation processes have been developed.

Process 0

In addition to the local law Process 0 uses the static averaging and the kinematic localisation operators A_0^s and L_0^k. In this case there is no micromechanical parameter to introduce. For an isotropic case, the analytical determination of the elastic constants is possible because of the simple shape of some of these relations:

First

$$P(\mathbf{n}) = \frac{1}{4\pi} \tag{19}$$

with

$$\boldsymbol{\sigma}_{ij} = \boldsymbol{\sigma}_0\boldsymbol{\delta}_{ij} \quad \text{the contact forces are} \quad \mathbf{F}_n = \frac{3\boldsymbol{\sigma}_0}{N\bar{D}} \quad \mathbf{F}_t = 0. \tag{20}$$

As a consequence:

$$\frac{\dot{\mathbf{F}}_n}{\dot{\mathbf{u}}_n} = k_n = \left(\frac{\sqrt{3R}G_m}{1-\nu_m}\right)^{2/3}\left(\frac{3\boldsymbol{\sigma}_0}{N\bar{D}}\right)^{1/3}\frac{4\pi\bar{D}}{3} = \text{constant with respect to } \mathbf{n} \tag{21}$$

$$\frac{\dot{\mathbf{F}}_t}{\dot{\mathbf{u}}_t} = k_t = \left(\frac{\sqrt{3R}G_m}{1-\nu_m}\right)^{2/3}\frac{2(1-\nu_m)}{2-\nu_m}\left(\frac{3\boldsymbol{\sigma}_0}{N\bar{D}}\right)^{1/3}\frac{4\pi\bar{D}}{3} = \text{constant with respect to } \mathbf{n}. \tag{22}$$

Should the medium be loaded incrementally with $\dot{\varepsilon}_{ij}$, the variation of the contact force distribution will be:

$$\dot{\mathbf{f}}(\mathbf{n}) = \dot{\mathbf{F}}(\mathbf{n})\frac{N\bar{D}}{3} = \frac{N\bar{D}}{3}(\dot{F}_n \cdot \mathbf{n} + \dot{\mathbf{F}}_t)$$

$$= \frac{N\bar{D}}{3}(k_t \dot{\mathbf{u}}_n \cdot \mathbf{n} + k_t \dot{\mathbf{u}}_t) = \frac{N\bar{D}}{4\pi}[k_n \mathbf{n}\dot{\varepsilon}\mathbf{n} \cdot \mathbf{n} + k_t(\dot{\varepsilon}\mathbf{n} - \mathbf{n}\dot{\varepsilon}\mathbf{n} \cdot \mathbf{n})].$$

Using A_0^s, the average stress increment resulting from the strain loading is:

$$\boldsymbol{\sigma} = \frac{3}{4\pi}\oint_{\text{Unit Sphere}} \dot{\mathbf{f}} \otimes \mathbf{n}\, d\Omega$$

$$= \frac{3}{4\pi}\oint_{\text{Unit Sphere}} \frac{N\bar{D}}{4\pi}[k_n \mathbf{n}\dot{\varepsilon}\mathbf{n} \cdot \mathbf{n} + k_t[\dot{\varepsilon}\mathbf{n} - \mathbf{n}\dot{\varepsilon}\mathbf{n} \cdot \mathbf{n}]] \otimes \mathbf{n}\, d\Omega$$

$$= \left(\frac{3G_m}{1-\nu_m}\frac{N\bar{D}^3}{\sqrt{2}}\right)^{2/3} \frac{1}{4\pi}\oint_{\text{Unit Sphere}} [\mathbf{n}\dot{\varepsilon}\mathbf{n} \cdot \mathbf{n} + \alpha(\dot{\varepsilon}\mathbf{n} - \mathbf{n}\dot{\varepsilon}\mathbf{n} \cdot \mathbf{n})] \otimes \mathbf{n}\, d\Omega$$

$$= \left(\frac{3G_m}{1-\nu_m}\frac{N\bar{D}^3}{\sqrt{2}}\right)^{2/3} (\boldsymbol{\sigma}_0)^{1/3}\left[\frac{1}{15}(1-\alpha)(\boldsymbol{\delta}_{ij}\operatorname{tr}\dot{\varepsilon} + 2\dot{\varepsilon}_{ij}) + \frac{\alpha}{3}\dot{\varepsilon}_{ij}\right] \quad \text{where } \alpha = \frac{2(1-\nu_m)}{2-\nu_m}$$

$$= \left(\frac{G_m}{1-\nu_m}\frac{N\bar{D}^3}{\sqrt{6}}\right)^{2/3} (\boldsymbol{\sigma}_0)^{1/3}\left(\frac{2+3\alpha}{5}\dot{\varepsilon}_{ij} + \boldsymbol{\delta}_{ij}\operatorname{tr}\dot{\varepsilon}\frac{1-\alpha}{5}\right)$$

$$= E_0\left(\frac{2+3\alpha}{5}\dot{\varepsilon}_{ij} + \boldsymbol{\delta}_{ij}\operatorname{tr}\dot{\varepsilon}\frac{1-\alpha}{5}\right) \quad \text{where } E_0 = \left(\frac{G_m N\bar{D}^3}{\sqrt{6}(1-\nu_m)}\right)^{2/3}\boldsymbol{\sigma}_0^{1/3}.$$

With the classical notations for the elastic characteristics (i.e. Young's modulus, Poisson's ratio, shear modulus and bulk modulus), the results of the first homogenisation process are:

$$E = E_0\frac{5-4\nu_m}{5-3\nu_m}, \quad \nu = \frac{\nu_m}{10-6\nu_m}, \quad G = E_0\frac{10-8\nu_m}{10-5\nu_m}, \quad 3K = E_0. \tag{23}$$

Process 1

Process 1 is obtained with the static localisation and the kinematic averaging operators L_1^s and A_1^k in the isotropic case, L_1^{se} and A_1^{ke} in the anisotropic one. The micromechanical parameter μ has to be considered in this process and the global elastic characteristics are for the isotropic case:

$$E = E_0\frac{1-\nu_m}{2(1-\nu_m)(1-\mu) + \frac{\mu^2}{5}(5-4\nu_m)}, \quad \nu = \frac{2(1-\nu_m)(5-10\mu) + 2\mu^2(5-4\nu_m)}{2(1-\nu_m)(20-20\mu) + 4\mu^2(5-4\nu_m)}$$

$$G = E_0\frac{10(1-\nu_m)}{(1-\nu_m)(25-30\mu) + 3\mu^2(5-4\nu_m)}, \quad 3K = E_0. \tag{24}$$

For an anisotropic medium and/or an anisotropic stress state a numerical determination is necessary because distributions $\mathbf{F}_n(\mathbf{n})$ and $\mathbf{F}_t(\mathbf{n})$ are no longer constants, $\mathbf{e}$ has now a non-zero value and the integral corresponding to the kinematic averaging operator cannot be achieved analytically in the general case, only a numerical determination is possible and will be analysed further on. Nevertheless, with the assumption of small anisotropy $\mathbf{e}$, it is possible to achieve the theoretical calculations. The elastic law can be written as

$$\dot{\varepsilon}_{ij}=\frac{1}{20E_0}\left\{\begin{array}{l}\dot{\boldsymbol{\sigma}}_{ij}\left(50-60\mu+18\mu^2+6\mu^2\frac{2-\nu_{\rm m}}{1-\nu_{\rm m}}\right)+\boldsymbol{\delta}_{ij}\,{\rm tr}\,\dot{\boldsymbol{\sigma}}\left(-10+20\mu-6\mu^2-2\mu^2\frac{2-\nu_{\rm m}}{1-\nu_{\rm m}}\right)\\ +\mathbf{e}_{ij}\,{\rm tr}\,\dot{\boldsymbol{\sigma}}\frac{12}{35}\left[20-34\mu+6\mu^2+\frac{2-\nu_{\rm m}}{1-\nu_{\rm m}}(\mu^2-7\mu)\right]\\ +\boldsymbol{\delta}_{ij}\,{\rm tr}\,\dot{\boldsymbol{\sigma}}\,{\rm tr}\,\mathbf{e}^2\frac{24}{175}\left[24-6\mu+\frac{2-\nu_{\rm m}}{1-\nu_{\rm m}}(7-2\mu)\right]\\ +\boldsymbol{\delta}_{ij}\,{\rm tr}\,\mathbf{e}\dot{\boldsymbol{\sigma}}\frac{12}{35}\left[5-13\mu+6\mu^2+\frac{2-\nu_{\rm m}}{1-\nu_{\rm m}}(\mu^2-7\mu)\right]\\ -(\dot{\boldsymbol{\sigma}}\mathbf{e})_{ij}\frac{12}{5}\mu^2\frac{2-\nu_{\rm m}}{1-\nu_{\rm m}}+{\rm tr}\,\dot{\boldsymbol{\sigma}}\mathbf{e}^2_{ij}\frac{24}{25}\mu\frac{2-\nu_{\rm m}}{1-\nu_{\rm m}}\\ +\boldsymbol{\delta}_{ij}\,{\rm tr}\,\mathbf{e}\dot{\boldsymbol{\sigma}}\mathbf{e}\frac{12}{25}\mu\frac{2-\nu_{\rm m}}{1-\nu_{\rm m}}+\boldsymbol{\delta}_{ij}\,{\rm tr}\,\dot{\boldsymbol{\sigma}}\mathbf{e}^2\frac{12}{25}\mu\frac{2-\nu_{\rm m}}{1-\nu_{\rm m}}-\boldsymbol{\delta}_{ij}\,{\rm tr}\,\dot{\boldsymbol{\sigma}}\,{\rm tr}\,\mathbf{e}^3\frac{48}{125}\end{array}\right\}. \tag{25}$$

With an anisotropy **e** of the following form

$$\mathbf{e}=\begin{bmatrix} e & & \\ & -e/2 & \\ & & -e/2\end{bmatrix} \tag{26}$$

the results of Process 1 become:

$$\begin{bmatrix}\dot{\varepsilon}_{11}\\ \dot{\varepsilon}_{22}\\ \dot{\varepsilon}_{33}\\ \dot{\varepsilon}_{23}\\ \dot{\varepsilon}_{13}\\ \dot{\varepsilon}_{12}\end{bmatrix}=\begin{bmatrix}E'_{11} & E'_{12} & E'_{13} & & & \\ E'_{21} & E'_{22} & E'_{23} & & & \\ E'_{31} & E'_{32} & E'_{33} & & & \\ & & & E'_{44} & & \\ & & & & E'_{55} & \\ & & & & & E'_{66}\end{bmatrix}\begin{bmatrix}\dot{\sigma}_{11}\\ \dot{\sigma}_{22}\\ \dot{\sigma}_{33}\\ \dot{\sigma}_{23}\\ \dot{\sigma}_{13}\\ \dot{\sigma}_{12}\end{bmatrix}. \tag{27}$$

As the coefficients $\mathbf{E}'_{ij}$ are complex ones, their expressions are given in Appendix 1.

Process 2

In addition to the local law Process 2 uses the second kinematic and the static averaging operators $A_0^{\rm s}$ and $L_2^{\rm k}$ in the isotropic case, $A_0^{\rm s}$ and $L_2^{\rm ke}$ if not. As for Processes 0 and 1, it is easy to define analytically in the isotropic case the elastic constants of the granular medium:

$$E=E_0\frac{5-4\nu_{\rm m}-3(1-\nu_{\rm m})b+(5-4\nu_{\rm m})\frac{3b\mu}{5}}{5-3\nu_{\rm m}-(1-\nu_{\rm m})b+(5-4\nu_{\rm m})\frac{b\mu}{5}},\quad \nu=\frac{\nu_{\rm m}+2(1-\nu_{\rm m})b-(5-4\nu_{\rm m})\frac{2b\mu}{5}}{10-6\nu_{\rm m}-2(1-\nu_{\rm m})b+(5-4\nu_{\rm m})\frac{2b\mu}{5}},$$

$$G=E_0\frac{10-8\nu_{\rm m}-6(1-\nu_{\rm m})b+(5-4\nu_{\rm m})\frac{6b\mu}{5}}{10-5\nu_{\rm m}},\quad 3K=E_0. \tag{28}$$

In the case of an anisotropic medium or anisotropic stress state, only numerical evaluation of the elastic constants can be achieved for the same reasons as explained previously. Nevertheless for small anisotropies, the elastic law can be written:

$$\dot{\sigma}_{ij} = E_0 \begin{bmatrix} \frac{2}{5}\dot{\varepsilon}_{ij}\left\{1+\frac{3b\mu}{5}+3\frac{1-v_m}{2-v_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\} \\ +\frac{1}{5}\delta_{ij}\,\mathrm{tr}\,\dot{\varepsilon}\left\{1-\frac{2b\mu}{5}-2\frac{1-v_m}{2-v_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\} \\ +\delta_{ij}\,\mathrm{tr}\,\mathbf{e}\dot{\varepsilon}\frac{6}{175}\left\{1-7b+\frac{3b\mu}{5}-2\frac{1-v_m}{2-v_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\}-\frac{36}{625}b\,\mathrm{tr}\,\mathbf{e}\dot{\varepsilon}\mathbf{e}_{ij} \\ +\mathrm{tr}\,\dot{\varepsilon}\mathbf{e}_{ij}\frac{6}{175}\left\{1-\frac{4b\mu}{5}-2\frac{1-v_m}{2-v_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\} \\ +(\dot{\varepsilon}\mathbf{e})_{ij}\frac{12}{25}\frac{1-v_m}{2-v_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right] \end{bmatrix}. \tag{29}$$

With the same anisotropy as in Process 1, the results of Process 2 become

$$\begin{bmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{33} \\ \dot{\sigma}_{23} \\ \dot{\sigma}_{13} \\ \dot{\sigma}_{12} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & & & \\ E_{21} & E_{22} & E_{23} & & & \\ E_{31} & E_{32} & E_{33} & & & \\ & & & E_{44} & & \\ & & & & E_{55} & \\ & & & & & E_{66} \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_{11} \\ \dot{\varepsilon}_{22} \\ \dot{\varepsilon}_{33} \\ \dot{\varepsilon}_{23} \\ \dot{\varepsilon}_{13} \\ \dot{\varepsilon}_{12} \end{bmatrix}. \tag{30}$$

As the coefficients E_{ij} are complex ones, their expressions are given in Appendix 2.

3. ANALYSIS OF THE RESULTS IN THE ISOTROPIC CASE

Homogenisation results for granular media have been proposed in the literature based on the Voigt type localisation operator (Walton, 1987; Jenkins *et al.*, 1989; Chang and Misra, 1990). These approaches correspond to Process 0. In this section we will compare the three possible approaches with experimental results.

El Hosri (1984) using a triaxial device has applied small cycles of deviatoric loadings to samples submitted to different isotropic initial stresses. He supposed the medium isotropic and then can deduce from these tests the two isotropic elastic constants of the granular medium. The initial void ratio is 0.875. The elastic characteristics of the glass are $G_m = 3.0 \times 10^{10}$ Pa and $v_m = 0.2$. The three processes (i.e. Voigt type homogenisation, static localisation and second level of kinematic localisation) are compared in terms of Young's modulus and Poisson's ratio on Figs 2a–f.

Figures 2a, b show that the Voigt type homogenisation gives moduli stronger than the real ones and a totally unrealistic Poisson's ratio. Nevertheless, the variation of each elastic characteristic with the isotropic stress is relevant.

Figures 2c, d show that for the static localisation homogenisation the best results are obtained with a value of μ equal to approximately 0.0. This value is not relevant to that found in numerical simulations (Mahboubi, 1995) : $\mu = 0.7$ gives a better description of the contact force distribution.

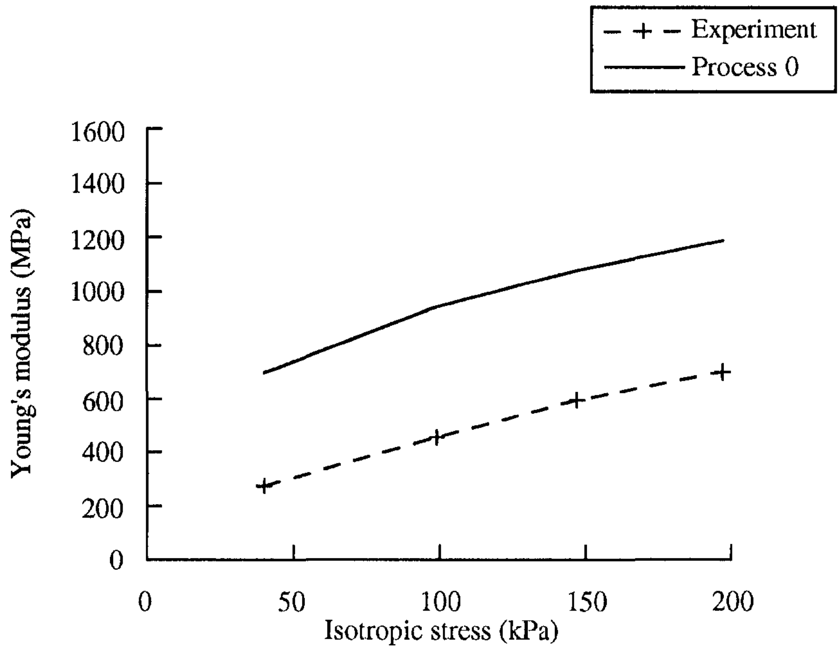
In Figs 2e, f the results obtained with Process 2 are shown in the case of constant values of μ and for different values of b. μ is taken equal to the value previously mentioned deduced from numerical simulations. The experimental results seem to be well explained with b around 3.5.

In Figs 3a, b, the results corresponding to the best values of b and μ are presented. Poisson's ratio is independent of the isotropic stress as for the experimental results and

Young's modulus varies as the n-power of the isotropic stress with $n = 1/3$ ($n = 0.59$ in experimental results).

It has been shown in previous works (Cambou *et al.*, 1994) that better results can be obtained with a decreasing value of b in Process 2. This can be explained by the fact that parameter b is related to the possibility of rotation and loss of contact of particles which is decreasing when the isotropic stress is increased. For the sake of simplicity and for a better understanding of the results in an anisotropic case (see Section 4.2), no variation of b has been taken into account.

(a)



(b)

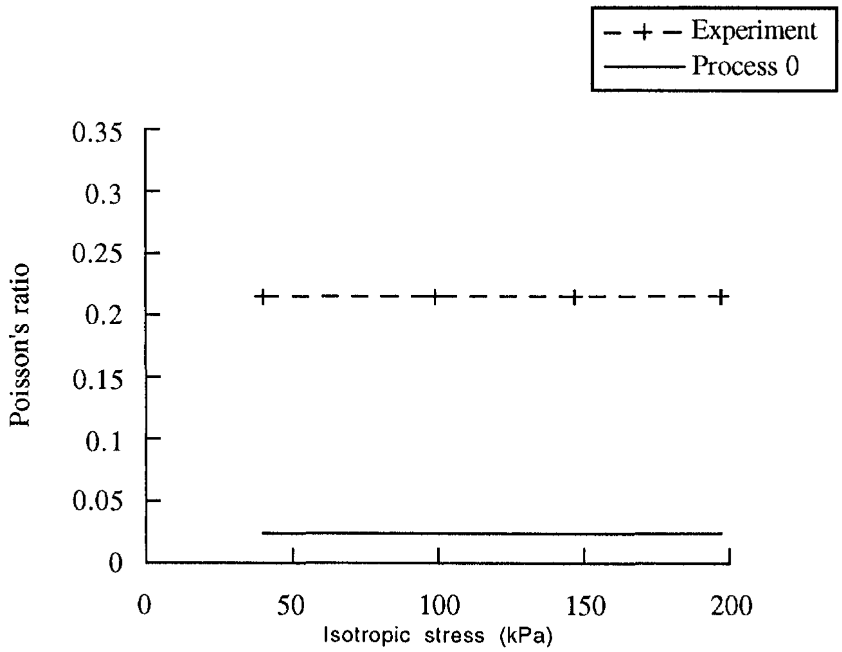
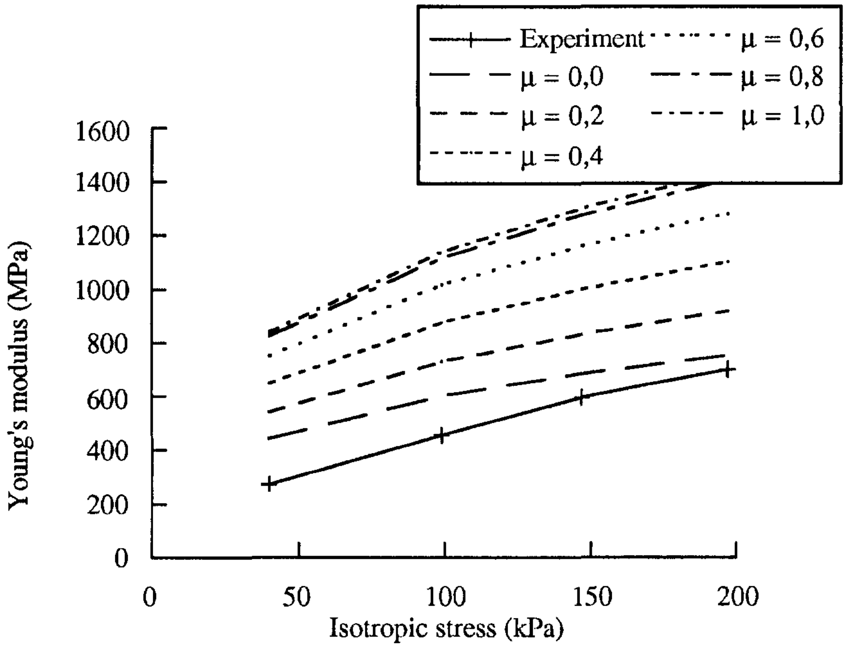


Fig. 2. (a) Comparison between experimental results and Process 0 homogenised law—Young's modulus. (b) Comparison between experimental results and Process 0 homogenised law—Poisson's ratio. (*Continued overleaf.*)

(c)



(d)

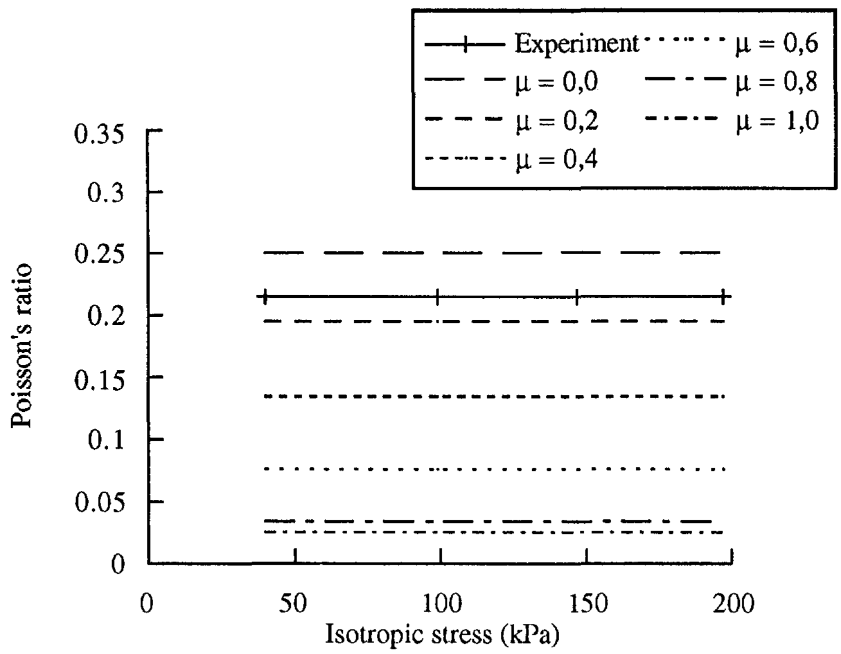


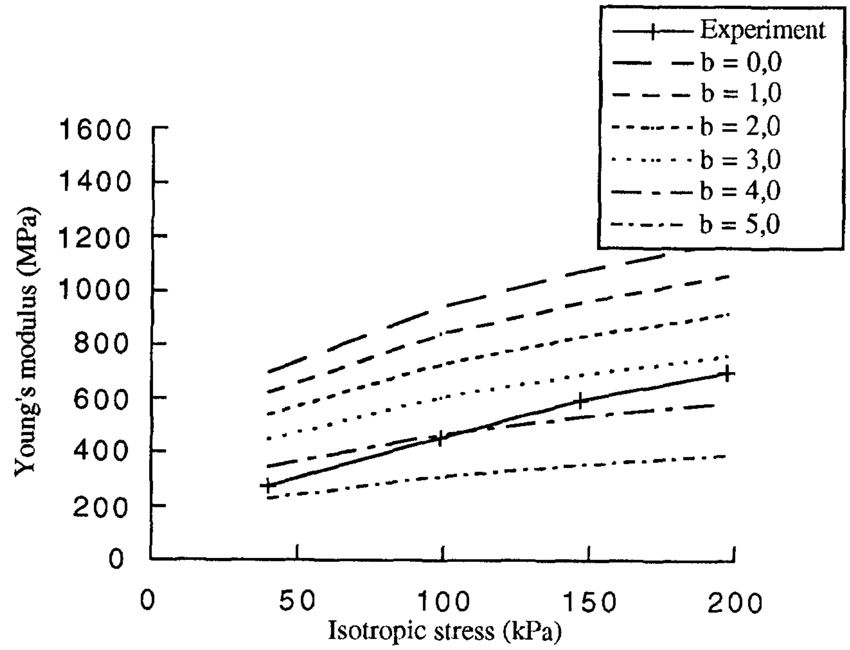
Fig. 2 (*Continued*). (c) Comparison between experimental results and Process 1 homogenised law—Young's modulus. (d) Comparison between experimental results and Process 1 homogenised law—Poisson's ratio.

4. ANALYSIS OF THE RESULTS IN ANISOTROPIC CASES

4.1. *Stress anisotropy*

All the previous results were established for an isotropic stress state assuming an isotropic medium. An anisotropic stress state is now considered still assuming an isotropic medium (i.e. e = 0). The elastic characteristics (i.e. coefficients E_{ij} of the elasticity matrix given in Section 2) cannot be derived analytically. A numerical calculation is needed to compare the micromechanical approach with experimental data obtained during a triaxial

(e)



(f)

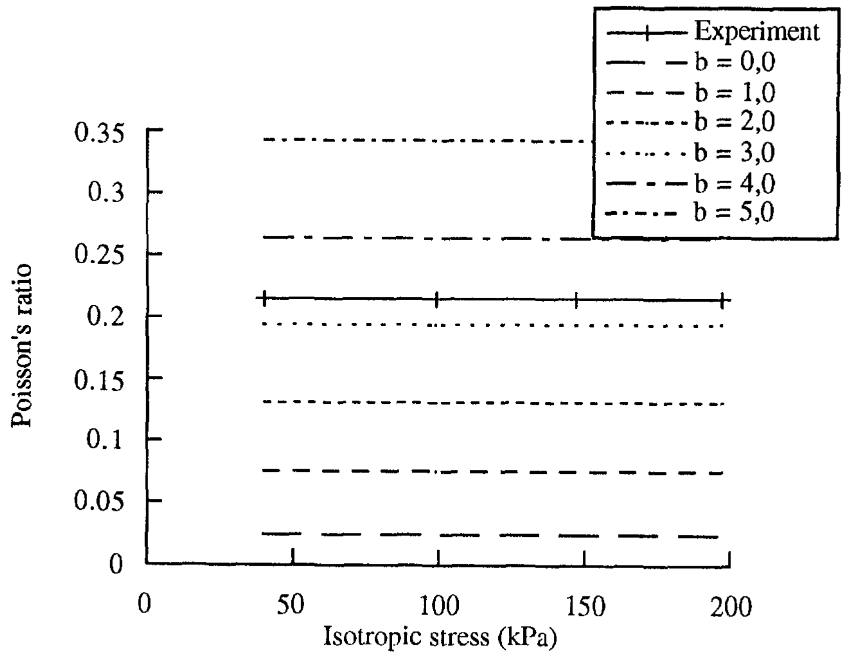
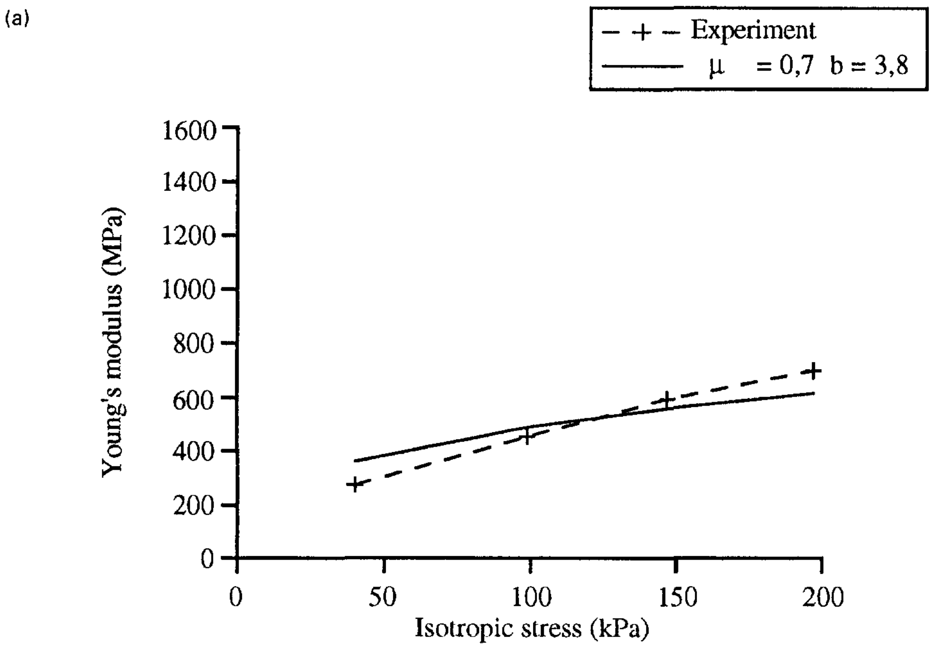


Fig. 2 (*Continued*). (e) Comparison between experimental results and Process 2 homogenised law—Young's modulus : $\mu = 0.7$. (f) Comparison between experimental results and Process 2 homogenised law—Poisson's ratio : $\mu = 0.7$.

compression test on glass ball assembly (Agarwal and Ishibashi, 1992). The different elastic characteristics are obtained from wave velocity measurements. The glass ball mixture (2 sizes with diameter of 0.215 mm and 0.256 mm) has an initial void ratio of 0.580. The initial isotropy of the granular medium can be checked in this case. The three constants E_{11}, E_{22}, E_{33} are initially equal.

If we assume that μ and b remain constant during the test, the description of measured elastic coefficients is good (Fig. 4). For small stress anisotropy, the assumption of isotropic medium is relevant. Most of the elastic anisotropy is due to the stress anisotropy itself. But



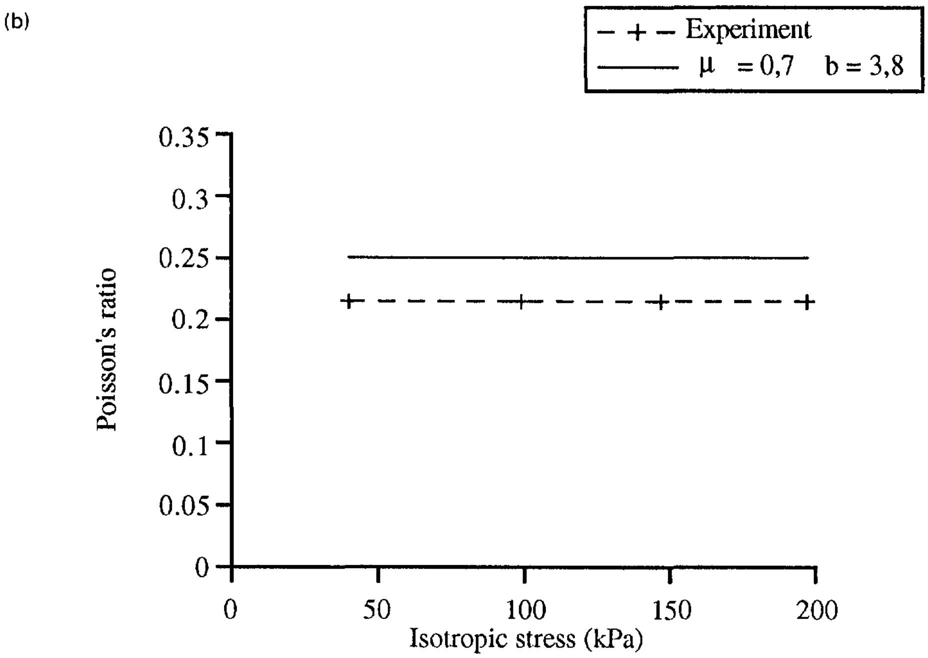


Fig. 3. (a) Comparison between experimental results and Process 2 homogenised law—Young's modulus: $\mu = 0.7$ $b = 3.8$. (b) Comparison between experimental results and Process 2 homogenised law—Poisson's ratio: $\mu = 0.7$ $b = 3.8$.

for great values of deviatoric stress, a structural anisotropy **e** is needed to explain the experimental results.

4.2. *Structural anisotropy*

In this section a simple comparison with experimental results of isotropic compression test on the same glass ball assembly previously mentioned with initial anisotropy is done (Agarwal and Ishibashi, 1992). The initial anisotropy of material can easily be checked as the two characteristics E_{11} and E_{22} are different. The medium anisotropy is here assumed to be as in eqn (26):

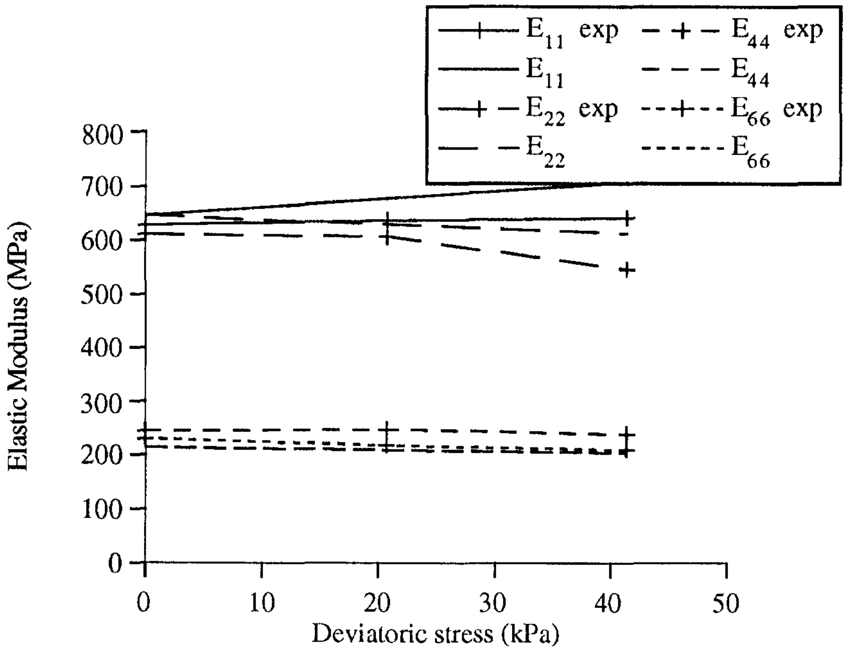


Fig. 4. Comparison of experimental compression results and Process 2 homogenised law with μ and b constant: $\mu = 0.7$, $b = 3.8$.

$$\mathbf{e} = \begin{bmatrix} e & & \\ & -e/2 & \\ & & -e/2 \end{bmatrix}.$$

This anisotropy can result if $e > 0$ (respectively, $e < 0$) from a compression (respectively, extension) test in the 1-direction.

For constant values of μ, $\mathbf{e}$ and b, the theoretical approach matches the experimental results (Fig. 5). These results should be considered very carefully because of the difficulty

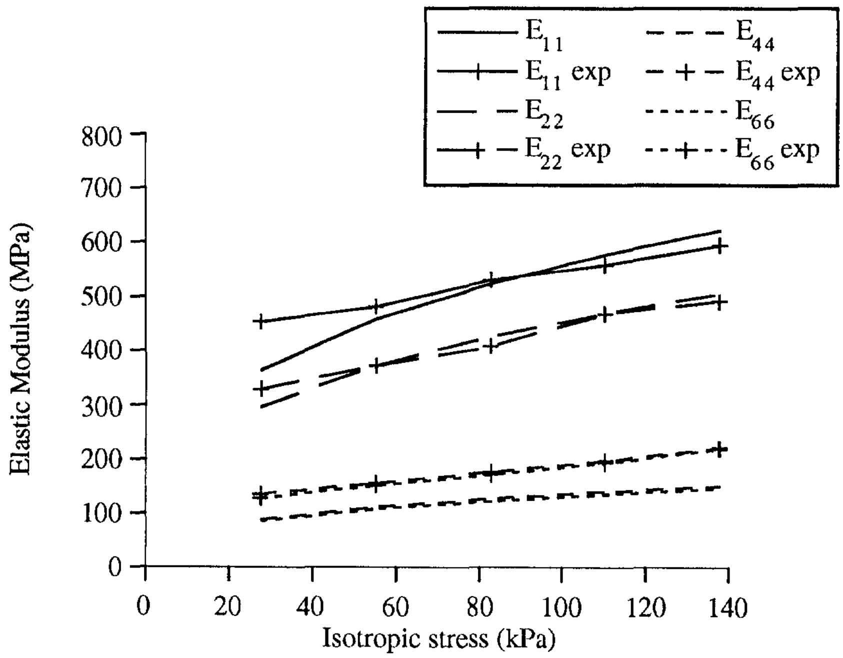


Fig. 5. Comparison of the experimental isotropic compression results and Process 2 homogenised law with μ, b, $\mathbf{e}$ constant: $\mu = 0.7$, $b = 4.78$, $\mathbf{e} = 0.1$.

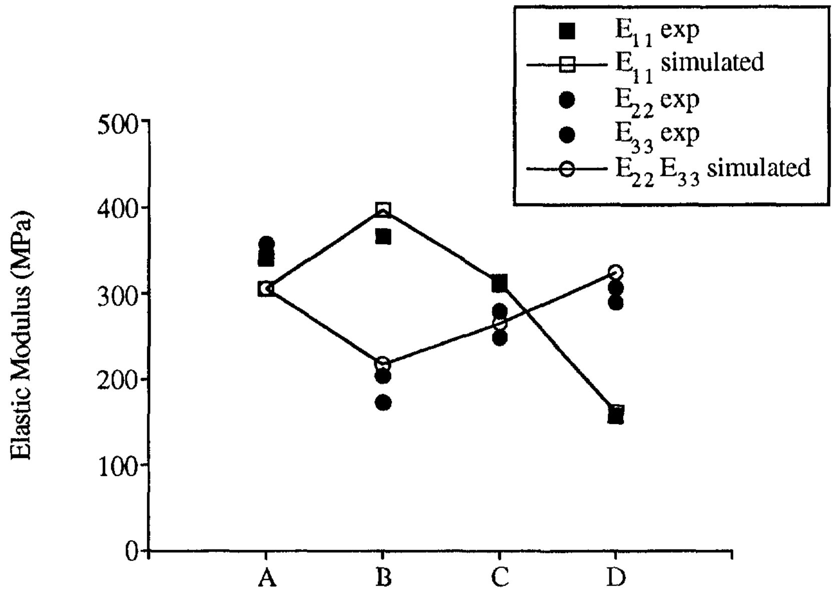


Fig. 6. Comparison of the numerical results (Cundall *et al.*, 1989) and Process 2 homogenised law with μ, b constant: $\mu = 0.5$, $b = 3.0$.

of measurement of the elastic characteristics of granular materials first and also because of the existence of a dissipative behaviour of granular medium even for small loadings. Thus, various authors (Luong, 1993) showed that for small load cycles, the dissipated work is relatively small and negligable below the characteristic threshold, but that, on the contrary, a large frictional energy has to be dissipated above the characteristic threshold because of the instability of a great number of contacts. So, for strong anisotropy, experimental results are not so reliable.

4.3. *Stress and structural anisotropy*

Usually, both anisotropies (stress and structure) have to be considered simultaneously. In this part, a comparison is done with numerical simulations on TRUBAL (Cundall *et al.*, 1989). The elastic characteristics E_{11}, E_{22} and E_{33} [cf. eqn (30)] are determined during a constant mean pressure triaxial test on a initially isotropic 432 sphere assembly. Point A corresponds to the initial isotropic stress state. Point B is the maximum compression point (compression in direction 1). Point C is the result of unloading from B to the isotropic stress state and finally point D corresponds to the final state of an extension test conducted from C.

Figure 6 shows the results of the numerical simulation and those of the homogenisation approach. For the four points A, B, C and D, μ is equal to 0.5 and b to 3.0. For point A, the medium is supposed isotropic so $\mathbf{e} = 0$. For point B, because of the development of a structural anisotropy in direction of compression, $\mathbf{e}$ is now equal to 0.1. This value of $\mathbf{e}$ has been defined from numerical simulations conducted by Mahboubi using the same numerical program (Mahboubi, 1995). For point C, the anisotropy $\mathbf{e}$ decreases to 0.05. Finally, for point D, the extension loading leads to $\mathbf{e} = -0.1$.

Once more, the simulated values of E_{11}, E_{22} and E_{33} seem to follow quite well the numerical results. As for experimental comparisons, the numerical results have to be considered carefully because of the necessary simplified numerical scheme used in the DEM.

5. CONCLUSION

The results presented in this paper are obtained considering a local elastic non-linear law. It seems that the description is good for either isotropic or anisotropic stress state in the case of an isotropic medium. A few results have been given in the case of an initial structural anisotropy induced by fabric or loading history. Comparisons with experimental and numerical results have shown the relevance of the micromechanical approach in modelling the anisotropic elastic behaviour of granular materials.

This elastic part of strain is especially important when granular media such as soils are submitted to vibrations or very small amplitude cycles. But as the non-reversible phenomena are important, as soon as the strain is greater than 10^{-4}, the present model has to be extended considering local non-reversible contact laws. In particular, the chosen contact law should be able to take into account the sliding between particles and the loss of contact. As a consequence, evolution laws should be proposed for the micromechanical parameters introduced, and especially for the anisotropy tensor **e**. These analyses have been recently achieved and will be described in further papers.

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APPENDIX

The results presented in Appendices 1 and 2 are those of anisotropic medium loaded isotropically and correspond to the application of Processes 1 and 2. The matrices $\mathbf{E}'_{ij}$ and $\mathbf{E}_{ij}$ have been defined previously (see Section 2).

Appendix 1

$$E_0E'_{11} = 2-2\mu+\frac{3}{5}\mu^2+\frac{1}{5}\mu^2\frac{2-\nu_m}{1-\nu_m}+\frac{3e}{175}\left[45-47\mu+12\mu^2-\frac{2-\nu_m}{1-\nu_m}(8\mu+5\mu^2)\right]$$
$$+\frac{3e^2}{875}\left[72-18\mu+\frac{2-\nu_m}{1-\nu_m}(21-22\mu)\right]-\frac{4e^3}{625}$$

$$E_0E'_{22} = E_0E'_{33} = 2-2\mu+\frac{3}{5}\mu^2+\frac{1}{5}\mu^2\frac{2-\nu_m}{1-\nu_m}-\frac{3e}{350}\left[45-47\mu+12\mu^2-\frac{2-\nu_m}{1-\nu_m}(8\mu+5\mu^2)\right]$$
$$+\frac{3e^2}{875}\left[72-18\mu+\frac{2-\nu_m}{1-\nu_m}(21-\mu)\right]-\frac{9e^3}{625}$$

$$E_0E'_{12} = E_0E'_{13} = -\frac{1}{2}+\mu-\frac{3}{10}\mu^2-\frac{1}{10}\mu^2\frac{2-\nu_m}{1-\nu_m}+\frac{3e}{350}\left[75-55\mu+6\mu^2+\frac{2-\nu_m}{1-\nu_m}(5\mu+\mu^2)\right]$$
$$+\frac{3e^2}{875}\left[72-18\mu+\frac{2-\nu_m}{1-\nu_m}\left(21+\frac{23}{2}\mu\right)\right]-\frac{9e^3}{625}$$

$$E_0E'_{21} = E_0E'_{31} = -\frac{1}{2}+\mu-\frac{3}{10}\mu^2-\frac{1}{10}\mu^2\frac{2-\nu_m}{1-\nu_m}+\frac{3e}{350}\left[-30+8\mu+6\mu^2+\frac{2-\nu_m}{1-\nu_m}(-13\mu+\mu^2)\right]$$
$$+\frac{3e^2}{875}\left[72-18\mu+\frac{2-\nu_m}{1-\nu_m}\left(21+\frac{23}{2}\mu\right)\right]-\frac{9e^3}{625}$$

$$E_0E'_{23} = E_0E'_{32} = -\frac{1}{2}+\mu-\frac{3}{10}\mu^2-\frac{1}{10}\mu^2\frac{2-\nu_m}{1-\nu_m}-\frac{3e}{350}\left[45-47\mu+12\mu^2+2\frac{2-\nu_m}{1-\nu_m}(-4\mu+\mu^2)\right]$$
$$+\frac{3e^2}{875}\left[72-18\mu+\frac{2-\nu_m}{1-\nu_m}(21-\mu)\right]-\frac{9e^3}{625}$$

$$E_0E'_{44} = \frac{5}{2}-3\mu+\frac{9}{10}\mu^2+\frac{3}{10}\mu^2\frac{2-\nu_m}{1-\nu_m}+\frac{3e}{50}\frac{2-\nu_m}{1-\nu_m}\mu^2$$

$$E_0E'_{55} = E_0E'_{66} = \frac{5}{2}-3\mu+\frac{9}{10}\mu^2+\frac{3}{10}\mu^2\frac{2-\nu_m}{1-\nu_m}-\frac{6e}{50}\frac{2-\nu_m}{1-\nu_m}\mu^2.$$

Keeping first-order terms in e and using the expression of the isotropic elastic characteristics E, G and ν, the anisotropic elastic constants become:

$$E'_{11} = \frac{1}{E^{\text{iso}}}+\frac{3e}{175E_0}\left[45-47\mu+12\mu^2-\frac{2-\nu_m}{1-\nu_m}(8\mu+5\mu^2)\right]$$

$$E'_{22} = E'_{33} = \frac{1}{E^{\text{iso}}}-\frac{3e}{350E_0}\left[45-47\mu+12\mu^2-\frac{2-\nu_m}{1-\nu_m}(8\mu+5\mu^2)\right]$$

$$E'_{12} = E'_{13} = -\frac{\nu^{\text{iso}}}{E^{\text{iso}}}+\frac{3e}{350E_0}\left[75-55\mu+6\mu^2+\frac{2-\nu_m}{1-\nu_m}(5\mu+\mu^2)\right]$$

$$E'_{21} = E'_{31} = -\frac{\nu^{\text{iso}}}{E^{\text{iso}}}+\frac{3e}{350E_0}\left[-30+8\mu+6\mu^2+\frac{2-\nu_m}{1-\nu_m}(-13\mu+\mu^2)\right]$$

$$E'_{23} = E'_{32} = -\frac{\nu^{\text{iso}}}{E^{\text{iso}}}-\frac{3e}{350E_0}\left[45-47\mu+12\mu^2+2\frac{2-\nu_m}{1-\nu_m}(-4\mu+\mu^2)\right]$$

$$E'_{44} = \frac{1}{G^{\text{iso}}}+\frac{3e}{50E_0}\frac{2-\nu_m}{1-\nu_m}\mu^2$$

$$E'_{55} = E'_{66} = \frac{1}{G^{\text{iso}}}-\frac{6e}{50E_0}\frac{2-\nu_m}{1-\nu_m}\mu^2.$$

Appendix 2

$$E_{11} = E_0\left(\frac{3}{5}+\frac{4b\mu}{25}+\frac{4}{5}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]+\frac{6e}{875}\left\{10-35b-b\mu+55\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\}-\frac{36be^2}{625}\right)$$

$$E_{22} = E_{33} = E_0\left(\frac{3}{5}+\frac{4b\mu}{25}+\frac{4}{5}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]-\frac{3e}{875}\left\{10-35b-b\mu+55\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\}-\frac{9be^2}{625}\right)$$

$$E_{12} = E_{13} = E_0\left(\frac{1}{5}-\frac{2b\mu}{25}-\frac{2}{5}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]+\frac{3e}{875}\left\{5+35b-11b\mu-15\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\}+\frac{18be^2}{625}\right)$$

$$E_{21} = E_{31} = E_0\left\{\frac{1}{5}-\frac{2b\mu}{25}-\frac{2}{5}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]+\frac{3e}{875}(5-70b+10b\mu)+\frac{18be^2}{625}\right\}$$

$$E_{23} = E_{32} = E_0\left(\frac{1}{5} - \frac{2b\mu}{25} - \frac{2}{5}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right] - \frac{3e}{875}\left\{10-35b-b\mu-15\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\} - \frac{9be^2}{625}\right)$$

$$E_{44} = E_0\left\{\frac{2}{5} + \frac{6b\mu}{25} + \frac{6}{5}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right] - \frac{6e}{25}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\}$$

$$E_{55} = E_{66} = E_0\left\{\frac{2}{5} + \frac{6b\mu}{25} + \frac{6}{5}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right] + \frac{12e}{25}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\}.$$

With the first order term in e and the elastic characteristics E, G and ν, these elastic constants become:

$$E_{11} = \frac{E^{iso}(1-\nu^{iso})}{(1-2\nu^{iso})(1+\nu^{iso})} + \frac{6eE_0}{875}\left\{10-35b-b\mu+55\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\}$$

$$E_{22} = E_{33} = \frac{E^{iso}(1-\nu^{iso})}{(1-2\nu^{iso})(1+\nu^{iso})} - \frac{3eE_0}{875}\left\{10-35b-b\mu+55\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\}$$

$$E_{12} = E_{13} = \frac{E^{iso}\nu^{iso}}{(1-2\nu^{iso})(1+\nu^{iso})} + \frac{3eE_0}{875}\left\{5+35b-11b\mu-15\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\}$$

$$E_{21} = E_{31} = \frac{E^{iso}\nu^{iso}}{(1-2\nu^{iso})(1+\nu^{iso})} + \frac{3eE_0}{875}(5-70b+10b\mu)$$

$$E_{23} = E_{32} = \frac{E^{iso}\nu^{iso}}{(1-2\nu^{iso})(1+\nu^{iso})} - \frac{3eE_0}{875}\left\{10-35b-b\mu-15\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]\right\}$$

$$E_{44} = \frac{E^{iso}}{1+\nu^{iso}} - \frac{6eE_0}{25}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right]$$

$$E_{55} = E_{66} = \frac{E^{iso}}{1+\nu^{iso}} + \frac{12eE_0}{25}\frac{1-\nu_m}{2-\nu_m}\left[1-b\left(1-\frac{3\mu}{5}\right)\right].$$